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ON THE CONVERGENCE OF AN ITERATION PROCESS FOR NONSELF TOTALLY ASYMPTOTICALLY I-QUASI-NONEXPANSIVE MAPPINGS

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ABSTRACT. In this paper, we first define nonself total asymptotically I-nonexpansive mappings and nonself total asymptotically I-quasi-nonexpansive mappings. Then, we prove weak and strong convergence theorems of a composite iterative process to a common fixed point of nonself total asymptotically quasi-nonexpansive mappings and nonself total asymptotically I-quasi-nonexpansive mappings, defined on a nonempty closed convex subset of uniformly convex Banach space.

1. Introduction

Fixed point theory itself is a beautiful mixture of analysis, topology and geometry. Over last few decades, fixed point theory has been revealed itself as a very powerful and important tool in the study of nonlinear phenomena. Fixed point techniques, in particular, have been applied with success in diverse fields such as in economics, biology, chemistry, engineering and technology, game theory and physics. Let E be a real Banach space with K its nonempty subset. Let $T : K \to K$ be a mapping. A point $x \in K$ is called a fixed point of T iff $Tx = x$. In this paper, $\mathbb{N}$ stands for the set of natural numbers. We will also denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in K : Tx = x\}$ and by $F := F(T) \cap F(I)$, the set of common fixed points of two mappings T and I. Let K be a nonempty closed subset of a real normed linear space X. Then

(1) T is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$.
(2) T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$, such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all $x, y \in K$ and all $n \in \mathbb{N}$.
(3) T is called uniformly L-Lipschitzian if there exists a real number $L > 0$ such that, $\|T^n x - T^n y\| \leq L\|x - y\|$ for all $n \in \mathbb{N}$ and all $x, y \in K$.

In 2003, Chidume, Ofoedu and Zegeye [3] further generalized the concept of asymptotically nonexpansive self-mapping, and proposed the concept of non-self asymptotically nonexpansive mapping, which is defined as follows:

Definition 1. *Let K be a nonempty subset of a real normed space E and let $P : E \to K$ be a nonexpansive retraction of E onto K.*

(1) Mapping $T : K \to E$ is called non-self asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$, such that $\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n\|x - y\|$ for all $x, y \in K$ and $n \in \mathbb{N}$.
(2) Mapping $T : K \to E$ is said to be a non-self asymptotically quasi-nonexpansive mapping if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$, such that $\|T(PT)^{n-1}x - p\| \leq k_n\|x - p\|$ for all $x, y \in K$ and $p \in F$.

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(3) Nonself mapping T is called uniformly L-Lipschitzian if there exists a real number $L > 0$ such that,, $\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L\|x - y\|$ for all $n \in \mathbb{N}$ and all $x, y \in K$.

It is easy to see that if T is an asymptotically nonexpansive mapping and $F(T) \neq \varnothing$, then it must be uniformly L-Lipschitzian as well as asymptotically quasi-nonexpansive, but the converse does not hold. Similarly, it is clear from the above definitions that an asymptotically nonexpansive nonself mapping must be nonself uniformly L-Lipschitzian as well as nonself asymptotically quasi-nonexpansive (if $F(T) \neq \varnothing$).

In 2006, Alber et al. [4] introduced a more general class of asymptotically nonexpansive mappings called total asymptotically nonexpansive mappings. Their aim was to unify various definitions of classes of mappings associated with the class of asymptotically nonexpansive mappings and to prove a general convergence theorems applicable to all these classes of nonlinear mappings.

Definition 2. *Let K be a nonempty closed subset of a real normed linear space X. A mapping $T : K \to K$ is called totally asymptotically nonexpansive if there exist nonnegative real sequences $\{\mu_n\}, \{\lambda_n\}$ with $\mu_n, \lambda_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(0) = 0$ such that*

$$\|T^n x - T^n y\| \leq \|x - y\| + \mu_n \phi(\|x - y\|) + \lambda_n, \text{for all } x, y \in K,\ n \geq 1. \tag{1.1}$$

Remark 1. *If $\phi(\lambda) = \lambda$, then 1.1 reduces to $\|T^n x - T^n y\| \leq (1 + \mu_n)\|x - y\| + \lambda_n, n \geq 1$. If $\phi(\lambda) = \lambda$ and $\lambda_n = 0$ for all $n \geq 1$, then total asymptotically nonexpansive mappings coincide with asymptotically nonexpansive mappings. In addition, $\mu_n = 0$ and $\lambda_n = 0$ for all $n \geq 1$, then total asymptotically nonexpansive mappings becomes nonexpansive mappings. If $\mu_n = 0$ and $\lambda_n = \sigma_n = \max\{0, a_n\}$, where $a_n = \sup(\|T^n x - T^n y\| - \|x - y\|)$ for all $n \geq 1$, then 1.1 reduces to $\|T^n x - T^n y\| \leq \|x - y\| + a_n$,which has been studied as mappings asymptotically nonexpansive in the intermediate sense. The class of mappings which are asymptotically nonexpansive in the intermediate sense was introduced by Bruck et al.* [1]. *It is known* [2] *that if K is a nonempty closed convex bounded subset of a real uniformly convex Banach spaces E and T is a self-mapping of K which is asymptotically nonexpansive in the intermediate sense, then T has a fixed point.*

Strong convergence theorems for iterative processes of finite family of total asymptotically nonexpansive mappings in Banach spaces have been studied by Chidume and Ofoudo [5, 6] and they defined nonself total asymptotically nonexpansive mappings. Hu and Yang [7] obtained strong convergence theorems for three nonself total asymptotically nonexpansive mappings in real Banach spaces.

Definition 3. [5]*Let K be a nonempty subset of E. Let $P : E \to K$ be the nonexpansive retraction of E onto K. A nonself mapping $T : K \to E$ is called total asymptotically nonexpansive if there exist nonnegative real sequences $\{\mu_n\}, \{\lambda_n\}$ with $\mu_n, \lambda_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(0) = 0$ such that*

$$\left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \leq \|x - y\| + \mu_n \phi(\|x - y\|) + \lambda_n, \text{for all } x, y \in K,\ n \geq 1. \tag{1.2}$$

Definition 4. *Let K be a nonempty subset of E and $P : E \to K$ be the nonexpansive retraction of E onto K and $F(T) \neq \varnothing$. Then the nonself mapping $T : K \to E$ is called total asymptotically quasi-nonexpansive if there exist nonnegative real sequences $\{\mu_n\}$, $\{\lambda_n\}$ with $\mu_n, \lambda_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(0) = 0$ such that for all $x, y \in K$ and $p \in F(T)$,*

$$\left\|T(PT)^{n-1}x - p\right\| \leq \|x - p\| + \mu_n \phi(\|x - p\|) + \lambda_n, n \geq 1. \tag{1.3}$$

Remark 2. *If $\phi(\lambda) = \lambda$, then (1.2) reduces to $\left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \leq (1 + \mu_n)\|x - y\| + \lambda_n$, $n \geq 1$. If $\phi(\lambda) = \lambda$ and $\lambda_n = 0$ for all $n \geq 1$, then nonself total asymptotically nonexpansive mappings coincide with nonself asymptotically nonexpansive mappings.*

On the other hand, in [8] an asymptotically I-nonexpansive mapping was introduced. Namely, let $T, I : K \to K$ be two mappings of a nonempty subset K of a real normed linear space X. Then T is said to be asymptotically I-nonexpansive if there exists a sequence $\{\lambda_n\}$ with $\lim_{n\to\infty} \lambda_n = 1$ such that $\|T^n x - T^n y\| \leq \lambda_n \|I^n x - I^n y\|$ for all $x, y \in K$ and $n \geq 1$. T is called asymptotically I-quasi-nonexpansive if $F(T) \neq \emptyset$ and for a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$, we have $\|T^n x - p\| \leq \lambda_n \|I^n x - p\|$ for all $x, y \in K$ and $p \in F$.

Recently, Yang and Xie [11] defined the concept of non-self asymptotically I-nonexpansive mapping as follows:

Definition 5. *Let K be a nonempty subset of a real normed space E and $P : E \to K$ be a nonexpansive retraction of E onto K. Let $T, I : K \to E$ be two mappings.*

(1) T is called nonself I-asymptotically nonexpansive if there exists sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$, such that $\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n \|I(PI)^{n-1}x - I(PI)^{n-1}y\|$ for all $x, y \in K$ and $n \in \mathbb{N}$.
(2) T is called nonself I-asymptotically quasi-nonexpansive if $F = F(T) \cap F(I) \neq \emptyset$ and $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$, such that $\|T(PT)^{n-1}x - p\| \leq k_n \|I(PI)^{n-1}x - p\|$ for all $x, y \in K$ and $p \in F$.
(3) T is said to be uniformly L-Lipschitzian if there exists $L > 0$ such that for all $x, y \in K$ and all positive integer n

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L \|I(PI)^{n-1}x - I(PI)^{n-1}y\|.$$

Remark 3. *From above definitions, it is easy to see that if $F(T)$ is nonempty, an I-asymptotically nonexpansive mapping must be I-asymptotically quasi-nonexpansive. But the converse does not hold.*

Recently, Mukhamedov and Saburov [9] introduced the the folowing iteration process for common fixed points of totally asymptotically I-nonexpansive mappings in Banach spaces. For arbitrary chosen $x_0 \in K$, $\{x_n\}$ is define as follows:

$$\begin{cases} x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T^n y_n \\ y_n = (1 - \beta_n) x_n + \beta_n I^n x_n \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$. And they give the following definition for totally asymptotically I-nonexpansive mappings.

Definition 6. *Let $T, I : K \to K$ be two mappings of a nonempty subset K of a real normed linear space X. A mapping $T : K \to K$ is called totally asymptotically $I-$nonexpansive if there exist nonnegative real sequences $\{\mu_n\}$, $\{\lambda_n\}$ with $\mu_n, \lambda_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(0) = 0$ such that for all $x, y \in K$,*

$$\|T^n x - T^n y\| \leq \|I^n x - I^n y\| + \mu_n \phi(\|I^n x - I^n y\|) + \lambda_n, n \geq 1. \tag{1.5}$$

Note that (1.5) reduces to (1.1) when $I = Id$ (Id is the identity mapping). If $\phi(\lambda) = \lambda$, then one gets $\|T^n x - T^n y\| \leq (1 + \mu_n) \|I^n x - I^n y\| + \lambda_n$, which is a generalization of the asymptotically I-nonexpansive mapping. If $\phi(\lambda) = \lambda$ and $\lambda_n = 0$ for all $n \geq 1$, then total asymptotically I-nonexpansive mappings coincide with asymptotically I-nonexpansive mappings.

But up to now, non-self totally asymptotically quasi-nonexpansive mappings and non-self totally I-asymptotically (quasi)nonexpansive mappings has not been defined by any authors. Therefore we define some new mappings as follows:

Definition 7. *Let K be a nonempty subset of E and $P : E \to K$ be the nonexpansive retraction of E onto K. Let $T, I : K \to E$ be two mappings. Then the mapping $T : K \to E$ is called totally asymptotically $I-$nonexpansive if there exist nonnegative real sequences*

$\{\mu_n\}$, $\{\lambda_n\}$ with $\mu_n, \lambda_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(0) = 0$ such that for all $x, y \in K$,

$$\left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \le \left\|I(PI)^{n-1}x - I(PI)^{n-1}y\right\| + \mu_n \phi\left(\left\|I(PI)^{n-1}x - I(PI)^{n-1}y\right\|\right) + \lambda_n. \tag{1.6}$$

Definition 8. *Let K be a nonempty subset of E and $P : E \to K$ be the nonexpansive retraction of E onto K .Let $T, I : K \to E$ be two mappings. Then the mapping $T : K \to E$ is called totally asymptotically $I-$nonexpansive if there exist nonnegative real sequences $\{\mu_n\}$, $\{\lambda_n\}$ with $\mu_n, \lambda_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(0) = 0$ such that for all $x \in K$ and $p \in F = F(T) \cap F(I) \neq \emptyset$,*

$$\left\|T(PT)^{n-1}x - p\right\| \le \left\|I(PI)^{n-1}x - p\right\| + \mu_n \phi\left(\left\|I(PI)^{n-1} - p\right\|\right) + \lambda_n, n \ge 1. \tag{1.7}$$

Different types iteration processes for computing fixed points of nonlinear mappings was studied by various authors (see [7, 8, 11, 12, 16, 17, 18]). Thianwan [12] considered a new iterative scheme for two nonself asymptotically nonexpansive mappings (called projection type Ishikawa iteration) as follows:

$$\begin{cases} x_{n+1} = P((1-\alpha_n)y_n + \alpha_n T_1 (PT_1)^{n-1} y_n), \\ y_n = P((1-\beta_n)x_n + \beta_n T_2 (PT_2)^{n-1} x_n), \quad n \in \mathbb{N}. \end{cases} \tag{1.8}$$

where $T_1, T_2 : K \to E$ are nonself asymptotically nonexpansive mappings and $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate real sequences in $[0, 1)$. We modified this iteration process for our case as follows:

Let K be a nonempty closed convex subset of a real Banach space E and $P : E \to K$ be a nonexpansive retraction of E onto K. Let $T : K \to E$ be a nonself totally asymptotically I-quasi-nonexpansive mapping and I be a nonself totally asymptotically quasi-nonexpansive mapping. Then for two given sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $[0, 1)$ and $x_0 \in K$, we shall consider the following iteration scheme:

$$\begin{cases} x_{n+1} = P((1-\alpha_n)y_n + \alpha_n T (PT)^{n-1} y_n), \\ y_n = P((1-\beta_n)x_n + \beta_n I (PI)^{n-1} x_n), \quad n \in \mathbb{N}. \end{cases} \tag{1.9}$$

2. **Preliminaries**

Let E be a Banach space with dimension $E \ge 2$. The modulus of E is the function $\delta_E(\varepsilon) : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \left\|\frac{1}{2}(x+y)\right\| : \|x\| = 1, \ \|y\| = 1, \ \varepsilon = \|x - y\|\right\}.$$

A Banach space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$.

A Banach space X is said to satisfy Opial's condition if, for any sequence $\{x_n\}$ in X, $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in X$ with $y \neq x$, where $x_n \rightharpoonup x$ means that $\{x_n\}$ converges weakly to x. It is well know from that all Hilbert spaces and all l_p spaces for $1 < p < \infty$ have this property. However, the L_p spaces do not have this property unless $p = 2$.

Let K be a nonempty closed subset of a real Banach space X and $T : K \to K$ a mapping. The mapping T said to be semicompact if for any bounded sequence $\{x_n\} \subset K$ with $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $p \in F$.

A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be demiclosed at p if whenever $\{x_n\}$ is sequence in $D(T)$ such that $\{x_n\}$ converges weakly to $x^* \in D(T)$ and $\{Tx_n\}$ converges strongly to p, then $Tx^* = p$.

Recall that the mapping $T : K \to K$ with $F(T) \neq \emptyset$ is said to satisfy condition (A) [13] if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0, \infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in K$, where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$. Khan and Fukharuddin [10] modified the condition (A) for two mappings as follows: Two mappings T_1, $T_2 : K \to K$ are said to satisfy condition (A') [10] if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0, \infty)$ such that

$$\frac{1}{2}\left(\|x - T_1 x\| + \|x - T_2 x\|\right) \geq f(d(x, F))$$

for all $x \in K$, where $d(x, F) = \inf\{\|x - p\| : p \in F := F(T_1) \cap F(T_2)\}$.

We need the following lemmas for our main results.

Lemma 1. [14] *If $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are two sequences of nonnegative real numbers such that*

$$a_{n+1} \leq (1 + b_n) a_n + c_n, \ n \geq 1.$$

Suppose that $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Then $\{a_n\}$ is bounded and $\lim_{n\to\infty} a_n$ exists.

Lemma 2. [15] *Let X be a uniformly convex Banach space and let $\{t_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences of X such that*

$$\limsup_{n\to\infty} \|x_n\| \leq d, \ \limsup_{n\to\infty} \|y_n\| \leq d \ \text{and} \ \lim_{n\to\infty} \|(1 - t_n) x_n + t_n y_n\| = d$$

hold for some $d \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

3. Main results

In this section, we prove convergence of the iteration sceheme (1.9) to common fixed point of a nonself total asymptotically I-quasi-nonexpansive mapping and nonself totally asymptotically quasi-nonexpansive mapping. We shall assume that $F = F(T) \cap F(I) \neq \emptyset$. In order to prove our main results, the following lemmas are needed.

Lemma 3. *Let E be a real Banach space, K be a nonempty subset of E which is also a nonexpansive retract with retraction P. Let $T : K \to E$ be a nonself totally asymptotically I-quasi-nonexpansive mappings and $I : K \to K$ be a nonself totally asymptotically quasi-nonexpansive mapping. Then there exist nonnegative real sequences $\{\mu_n\}, \{\lambda_n\}$ with $\mu_n, \lambda_n \to 0$ as $n \to \infty$ and strictly increasing continuous function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(0) = 0$ such that for all $x \in K$ and $x^* \in F$*

$$\begin{aligned} \left\|I(PI)^{n-1}x - x^*\right\| &\leq \|x - x^*\| + \mu_n \phi(\|x - x^*\|) + \lambda_n, \\ \left\|T(PT)^{n-1}x - x^*\right\| &\leq \left\|I(PI)^{n-1}x - x^*\right\| + \mu_n \phi\left(\left\|I(PI)^{n-1}x - x^*\right\|\right) + \lambda_n. \end{aligned} \tag{3.1}$$

Proof. Since $T : K \to K$ is a nonself totally asymptotically I-quasi-nonexpansive mappings and $I : K \to K$ is a totally asymptotically quasi-nonexpansive mapping. Then there exist nonnegative real sequences $\{\mu_n'\}, \{\lambda_n'\}$ and $\{\tilde{\mu}_n\}, \{\tilde{\lambda}_n\}, n \geq 1$ with $\mu_n', \lambda_n', \tilde{\mu}_n, \tilde{\lambda}_n \to 0$ $(n \to \infty)$ and there exist strictly increasing continuous functions $\phi_1, \phi_2 : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi_1(0) = \phi_2(0) = \phi_3(0) = 0$ such that for all $x, y \in K$,

$$\left\|I(PI)^{n-1}x - x^*\right\| \leq \|x - x^*\| + \tilde{\mu}_n \phi_1(\|x - x^*\|) + \tilde{\lambda}_n,$$

$$\left\|T(PT)^{n-1}x - p\right\| \leq \left\|I(PI)^{n-1}x - x^*\right\| + \mu_n' \phi_2\left(\left\|I(PI)^{n-1}x - x^*\right\|\right) + \lambda_n', n \geq 1.$$

Setting

$$\mu_n = \max\{\mu_n', \tilde{\mu}\}, \lambda_n = \max\{\lambda_n', \tilde{\lambda}_n\},$$
$$\phi(a) = \max\{\phi_1(a), \phi_2(a)\} \text{ for } a \geq 0,$$

then we obtain that there exist nonnegative real sequences $\{\mu_n\}$ and $\{\lambda_n\}$ with $\mu_n, \lambda_n \to 0$ as $n \to \infty$ and strictly increasing continuous function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(0) = 0$ such that for all $x \in K$ and $x^* \in F$,

$$\left\|I(PI)^{n-1}x - x^*\right\| \leq \|x - x^*\| + \mu_n \phi\left(\|x - x^*\|\right) + \lambda_n$$

$$\left\|T(PT)^{n-1}x - p\right\| \leq \left\|I(PI)^{n-1}x - x^*\right\| + \mu_n \phi\left(\left\|I(PI)^{n-1}x - x^*\right\|\right) + \lambda_n^{'}, n \geq 1.$$

This completes the proof. □

Lemma 4. *Let E be a real Banach space, K be a nonempty closed convex subset of E which is also a nonexpansive retract with retraction P. Let $T : K \to E$ be a nonself totally asymptotically I-quasi-nonexpansive mappings and $I : K \to K$ be a nonself totally asymptotically quasi-nonexpansive mapping with sequences $\{\mu_n\}, \{\lambda_n\}$ defined by (3.1) such that $\sum_{n=1}^{\infty}\mu_n < \infty, \sum_{n=1}^{\infty}\lambda_n < \infty$. Assume that there exist $M, M^* > 0$ such that $\phi(\lambda) \leq M^*\lambda$ for all $\lambda \geq M$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0, 1)$, $x^* \in F$ and $\{x_n\}$ is defined by (1.9). Then, $\{x_n\}$ is bounded and the limits $\lim_{n\to\infty} \|x_n - x^*\|$ and $\lim_{n\to\infty} d(x_n, F)$ exist, where $d(x_n, F) = \inf_{x^*\in F} \|x_n - p\|$.*

Proof. Let $x^* \in F$. It follows from (1.9) and (3.1) that

$$\begin{aligned}\|y_n - x^*\| &= \left\|P((1-\beta_n)x_n + \beta_n I(PI)^{n-1}x_n) - P(x^*)\right\| \\ &\leq (1-\beta_n)\|x_n - x^*\| + \beta_n \left\|I(PI)^{n-1}x_n - x^*\right\| \\ &\leq (1-\beta_n)\|x_n - x^*\| + \beta_n \left[\|x_n - p\| + \mu_n\phi\left(\|x_n - x^*\|\right) + \lambda_n\right] \\ &\leq \|x_n - x^*\| + \beta_n\mu_n\phi\left(\|x_n - x^*\|\right) + \lambda_n. \end{aligned} \quad (3.2)$$

Note that ϕ is an increasing function, it follows that $\phi(\lambda) \leq \phi(M)$ whenever $\lambda \leq M$ and (by hypothesis) $\phi(\lambda) \leq M^*\lambda$ if $\lambda \geq M$. In either case, we have

$$\phi(\lambda) \leq \phi(M) + M^*\lambda \quad (3.3)$$

for some $M, M^* \geq 0$. Thus, from (3.2) and (3.3), we have

$$\begin{aligned}\|y_n - x^*\| &\leq \|x_n - x^*\| + \beta_n\mu_n\left[\phi(M) + M^*\|x_n - x^*\|\right] + \lambda_n \\ &\leq (1 + M^*\mu_n)\|x_n - x^*\| + Q_1(\mu_n + \lambda_n) \end{aligned} \quad (3.4)$$

for some constant $Q_1 > 0$. Similarly, from (3.4) we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \left\|P((1-\alpha_n)y_n + \alpha_n T(PT)^{n-1}y_n - P(x^*)\right\| \\
&\le (1-\alpha_n)\|y_n - x^*\| + \alpha_n \left\|T(PT)^{n-1}y_n - x^*\right\| \\
&\le (1-\alpha_n)\|y_n - x^*\| + \alpha_n[\left\|I(PI)^{n-1}y_n - x^*\right\| \\
&+ \mu_n\phi\left(\left\|I(PI)^{n-1}y_n - x^*\right\|\right) + \lambda_n] \\
&\le (1-\alpha_n)\|y_n - x^*\| + \alpha_n[\left\|I(PI)^{n-1}y_n - x^*\right\| \\
&+ \mu_n\phi(M) + \mu_n M^*\left\|I(PI)^{n-1}y_n - x^*\right\| + \lambda_n] \\
&\le (1-\alpha_n)\|y_n - x^*\| + \alpha_n[(1+\mu_n M^*)\left\|I(PI)^{n-1}y_n - x^*\right\| \\
&+ \mu_n\phi(M) + \lambda_n] \\
&\le (1-\alpha_n)\|y_n - x^*\| + \alpha_n[(1+\mu_n M^*)(\|y_n - x^*\| \\
&+ \mu_n\phi(\|y_n - x^*\|) + \lambda_n) + \mu_n\phi(M) + \lambda_n] \\
&\le (1-\alpha_n)\|y_n - x^*\| + \alpha_n[(1+\mu_n M^*)(\|y_n - x^*\| + \mu_n\phi(M) \\
&+ \mu_n M^*\|y_n - x^*\| + \lambda_n) + \mu_n\phi(M) + \lambda_n] \\
&\le (1-\alpha_n)\|y_n - x^*\| + \alpha_n[(1+\mu_n M^*)((1+\mu_n M^*)\|y_n - x^*\| \\
&+ \mu_n\phi(M) + \lambda_n) + \mu_n\phi(M) + \lambda_n] \\
&\le (1-\alpha_n)\|y_n - x^*\| + \alpha_n[(1+2\mu_n M^* + (\mu_n M^*)^2)\|y_n - x^*\| \\
&+ (1+M^*\mu_n)(\mu_n\phi(M) + \lambda_n) + \mu_n\phi(M) + \lambda_n] \\
&\le \|y_n - x^*\| + (2\mu_n M^* + (\mu_n M^*)^2)\|y_n - x^*\| \\
&+ (1+M^*\mu_n)(\mu_n\phi(M) + \lambda_n) + \mu_n\phi(M) + \lambda_n \\
&\le (1+2\mu_n M^* + (\mu_n M^*)^2)\|y_n - x^*\| \\
&+ (1+M^*\mu_n)(\mu_n\phi(M) + \lambda_n) + \mu_n\phi(M) + \lambda_n \\
&\le (1+2\mu_n M^* + (\mu_n M^*)^2)[(1+M^*\mu_n)\|x_n - x^*\| + Q_1(\mu_n + \lambda_n)] \\
&+ (1+M^*\mu_n)(\mu_n\phi(M) + \lambda_n) + \mu_n\phi(M) + \lambda_n \\
&\le (1+3\mu_n^2 M^* + 3\mu_n(M^*)^2 + (\mu_n M^*)^3)\|x_n - x^*\| \\
&+ (1+M^*\mu_n)(\mu_n\phi(M) + \lambda_n) + \mu_n\phi(M) + \lambda_n \\
&\le (1+M_2\mu_n)\|x_n - x^*\| + Q_2(\mu_n + \lambda_n) \qquad (3.5)
\end{aligned}$$

for some constants $M_2, Q_2 > 0$. Since (3.5) is true for each x^* in F. This implies that

$$d(x_{n+1}, x^*) \le (1+M_2\mu_n)\,d(x_n, x^*) + Q_2(\mu_n + \lambda_n). \qquad (3.6)$$

From Lemma 1, we obtain that $\lim_{n\to\infty}\|x_n - x^*\|$ and $\lim_{n\to\infty} d(x_n, F)$ exist. This completes the proof. □

Theorem 1. *Let E be a real Banach space, K be a nonempty closed convex subset of E which is also a nonexpansive retract with retraction P. Let $T : K \to E$ be a nonself uniformly L_1-Lipshitzian totally asymptotically I-quasi-nonexpansive mapping and $I : K \to K$ be a nonself uniformly L_2-Lipshitzian totally asymptotically quasi-nonexpansive continuous mapping with sequences $\{\mu_n\}, \{\lambda_n\}$ defined by (3.1) such that $\sum_{n=1}^{\infty}\mu_n < \infty, \sum_{n=1}^{\infty}\lambda_n < \infty$. Assume that there exist $M, M^* > 0$ such that $\phi(\lambda) \le M^*\lambda$ for all $\lambda \ge M$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0,1)$, $F \ne \emptyset$ and $\{x_n\}$ is defined by (1.9). Then the sequence $\{x_n\}$ converges strongly to a common fixed point of T and I if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.*

Proof. The necessity of the conditions is obvious. Thus, we need only prove the sufficiency. Suppose that $\liminf_{n\to\infty} d(x_n, F) = 0$. From Lemma 4, $\lim_{n\to\infty} d(x_n, F)$ exists. But by hypothesis $\liminf_{n\to\infty} d(x_n, F) = 0$, therefore we must have $\lim_{n\to\infty} d(x_n, F) = 0$.

Next we show that $\{x_n\}$ is a Cauchy sequence. As $1 + t \le \exp(t)$ for all $t > 0$, from (3.6), we obtain

$$\|x_{n+1} - x^*\| \le \exp(M_2\mu_n)(\|x_n - x^*\| + Q_2(\mu_n + \lambda_n)). \tag{3.7}$$

Thus, for any given m, n, iterating (3.7), we obtain

$$\begin{aligned}\|x_{n+m} - x^*\| &\le \exp(M_2\mu_{n+m-1})(\|x_{n+m-1} - x^*\| + Q_2(\mu_{n+m-1} + \lambda_{n+m-1}))\\ &\vdots\\ &\le \exp(\sum_{i=n}^{n+m-1} M_2\mu_i)\left(\|x_n - x^*\| + \sum_{i=n}^{n+m-1} Q_2(\mu_i + \lambda_i)\right)\\ &\le \exp(\sum_{i=n}^{\infty} M_2\mu_i)\left(\|x_n - x^*\| + \sum_{i=n}^{\infty} Q_2(\mu_i + \lambda_i)\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\|x_{n+m} - x_n\| &\le \|x_{n+m} - x^*\| + \|x_n - x^*\|\\ &\le \left[1 + \left(\exp(\sum_{i=n}^{\infty} M_2\mu_i)\right)\right]\|x_n - x^*\|\\ &\quad + \exp(\sum_{i=n}^{\infty} M_2\mu_i)\left(\sum_{i=n}^{\infty} Q_2(\mu_i + \lambda_i)\right).\end{aligned}$$

This imply that

$$\|x_{n+m} - x_n\| \le D\|x_n - x^*\| + D\left(\sum_{i=n}^{\infty}(\mu_i + \lambda_i)\right) \tag{3.8}$$

for same constant $D > 0$. Taking infimum over $x^* \in F$ in (3.8) gives

$$\|x_{n+m} - x_n\| \le Dd(x_n, F) + D\left(\sum_{i=n}^{\infty}(\mu_i + \lambda_i)\right)$$

Now, since $\lim_{n\to\infty} d(x_n, F) = 0$ and $\sum_{i=n}^{\infty}(\mu_i + \lambda_i) < \infty$, given $\epsilon > 0$, there exists an integer $N_1 > 0$ such that for all $n \ge N_1$, $d(x_n, F) < \epsilon/2D$ and $\sum_{i=n}^{\infty}(\mu_i + \lambda_i) < \epsilon/2D$. Consequently, from last inequality we have $\|x_{n+m} - x_n\| < \epsilon$, which means that $\{x_n\}$ is a Cauchy sequence in E, and completeness of E yield the existence of $q \in E$ such that $x_n \to q$. We now show that $q \in F$. Since T and I are continuous mappings, $F(T)$ and $F(I)$ are closed. Hence $F = F(T) \cap F(I)$ is a nonempty closed set. Suppose that $q \notin F$. Since F closed subset of E, we have that $d(q, F) > 0$. But, for all $x^* \in F$, we have

$$\|x^* - q\| \le \|x^* - x_n\| + \|x_n - q\|.$$

This implies

$$d(x^*, F) \le \|x^* - x_n\| + d(x_n, F),$$

so we obtain $d(x^*, F) = 0$ as $n \to \infty$, which contradicts $d(x^*, F) > 0$. Hence, q is a common fixed point of T, S and I. This completes the proof. □

For our next theorems, we start by proving the following lemma which will be needed in the sequel.

Lemma 5. *Let E be a uniformly convex real Banach space, K be a nonempty closed convex subset of E which is also a nonexpansive retract with retraction P. Let $T : K \to E$*

be a nonself uniformly L_1-Lipshitzian totally asymptotically I-quasi-nonexpansive mappings and $I : K \to K$ be a nonself uniformly L_2-Lipshitzian totally asymptotically quasi-nonexpansive mapping with sequences $\{\mu_n\}, \{\lambda_n\}$ defined by (3.1) such that $\sum_{n=1}^{\infty}\mu_n < \infty, \sum_{n=1}^{\infty}\lambda_n < \infty$. Assume that there exist $M, M^* > 0$ such that $\phi(\lambda) \leq M^*\lambda$ for all $\lambda \geq M$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $[\varepsilon, 1-\varepsilon]$, for some $\varepsilon \in (0,1)$. Suppose that $\{x_n\}$ is genareted iteratively by (1.9). Then

$$\lim_{n\to\infty}\|x_n - Tx_n\| = \lim_{n\to\infty}\|x_n - Ix_n\| = 0.$$

Proof. For any given $x^* \in F$, by Lemma 4, we know that $\lim_{n\to\infty}\|x_n - x^*\|$ exists. Assume $\lim_{n\to\infty}\|x_n - x^*\| = d$, for some $d \geq 0$. From (3.4) and (3.5), we get

$$\|y_n - x^*\| \leq (1 + M^*\mu_n)\|x_n - x^*\| + Q_1(\mu_n + \lambda_n) \tag{3.9}$$

and

$$\begin{aligned}\left\|T(PT)^{n-1}y_n - x^*\right\| &= \left\|I(PI)^{n-1}y_n - x^*\right\| + \mu_n\phi\left(\left\|I(PI)^{n-1}y_n - x^*\right\|\right) + \lambda_n \\ &\leq \left\|I(PI)^{n-1}y_n - x^*\right\| + \mu_n\phi(M) \\ &+ \mu_n M^*\left\|I(PI)^{n-1}y_n - x^*\right\| + \lambda_n \\ &\leq (1 + \mu_n M^*)\left\|I(PI)^{n-1}y_n - x^*\right\| + \mu_n\phi(M) + \lambda_n \\ &\leq (1 + \mu_n M^*)(\|y_n - x^*\| + \mu_n\phi(\|y_n - x^*\|) + \lambda_n) \\ &+ \mu_n\phi(M) + \lambda_n \\ &\leq (1 + \mu_n M^*)(\|y_n - x^*\| + \mu_n M^*\|y_n - x^*\| \\ &+ \mu_n\phi(M) + \lambda_n) + \mu_n\phi(M) + \lambda_n \\ &\leq (1 + \mu_n M^*)((1 + \mu_n M^*)\|y_n - x^*\| \\ &+ \mu_n\phi(M) + \lambda_n) + \mu_n\phi(M) + \lambda_n.\end{aligned} \tag{3.10}$$

Taking lim sup in the inequalities (3.9) and (3.10), we get

$$\limsup_{n\to\infty}\|y_n - x^*\| \leq d \tag{3.11}$$

and

$$\limsup_{n\to\infty}\left\|T(PT)^{n-1}y_n - x^*\right\| \leq \limsup_{n\to\infty}\|y_n - x^*\| \leq d. \tag{3.12}$$

Also,by using (1.9) we obtain that

$$\begin{aligned}d = \lim_{n\to\infty}\|x_n - x^*\| &= \lim_{n\to\infty}\|x_{n+1} - x^*\| \\ &= \lim_{n\to\infty}\left\|P((1-\alpha_n)y_n + \alpha_n T(PT)^{n-1}y_n - P(x^*)\right\| \\ &= \lim_{n\to\infty}\left\|(1-\alpha_n)(y_n - x^*) + \alpha_n(T(PT)^{n-1}y_n - x^*)\right\|.\end{aligned} \tag{3.13}$$

Now using (3.11) with (3.12) and applying Lemma to (3.13) one can finds

$$\lim_{n\to\infty}\left\|y_n - T(PT)^{n-1}y_n\right\| = 0 \tag{3.14}$$

On the other hand, from (3.1) we get that

$$\begin{aligned}\left\|I(PI)^{n-1}x_n - x^*\right\| &= \|x_n - x^*\| + \mu_n\phi(\|x_n - x^*\|) + \lambda_n \\ &\leq \|x_n - x^*\| + \mu_n M^*\|x_n - x^*\| + \mu_n\phi(M) + \lambda_n.\end{aligned} \tag{3.15}$$

Taking lim sup in the last inequalities, we have

$$\limsup_{n\to\infty}\left\|I(PI)^{n-1}x_n - x^*\right\| \leq \limsup_{n\to\infty}\|x_n - x^*\| \leq d \tag{3.16}$$

By (1.9) we get that

$$\begin{aligned}\|x_{n+1}-x^*\| &= \left\|P((1-\alpha_n)y_n+\alpha_n T(PT)^{n-1}y_n-P(x^*)\right\| \\ &\le (1-\alpha_n)\|y_n-x^*\|+\alpha_n\left\|T(PT)^{n-1}y_n-x^*\right\| \\ &\le (1-\alpha_n)\|y_n-x^*\|+\alpha_n\left\|T(PT)^{n-1}y_n-y_n\right\|+\alpha_n\|y_n-x^*\| \\ &\le \|y_n-x^*\|+\alpha_n\left\|T(PT)^{n-1}y_n-y_n\right\|. \end{aligned} \tag{3.17}$$

Putting $\lim_{n\to\infty}\|x_{n+1}-x^*\|=d$ in (3.17), we get

$$d\le \lim\inf_{n\to\infty}\|y_n-x^*\| \tag{3.18}$$

From (3.11) and (3.18), we obtain

$$\lim_{n\to\infty}\|y_n-x^*\|=d \tag{3.19}$$

It follows from (3.19) and (3.4)

$$\begin{aligned} d=\lim_{n\to\infty}\|y_n-x^*\| &\le \lim_{n\to\infty}\left\|(1-\beta_n)(x_n-x^*)+\beta_n(I(PI)^{n-1}x_n-x^*)\right\| \\ &\le \lim_{n\to\infty}(1+M^*\mu_n)\|x_n-x^*\|+Q_1(\mu_n+\lambda_n)\le \lim_{n\to\infty}\|x_n-x^*\|=d \end{aligned}$$

and so

$$\lim_{n\to\infty}\left\|(1-\beta_n)(x_n-x^*)+\beta_n(I(PI)^{n-1}x_n-x^*)\right\|=d \tag{3.20}$$

Using (3.16) with $\lim_{n\to\infty}\|x_n-x^*\|=d$ and applying Lemma 2 to (3.20) we get

$$\lim_{n\to\infty}\left\|x_n-I(PI)^{n-1}x_n\right\|=0 \tag{3.21}$$

From $y_n=P((1-\beta_n)x_n+\beta_n I(PI)^{n-1}x_n)$ and (3.21), we have

$$\begin{aligned}\|y_n-x_n\| &= \left\|P((1-\beta_n)x_n+\beta_n I(PI)^{n-1}x_n)-x_n\right\| \\ &\le \left\|(1-\beta_n)(x_n-x_n)+\beta_n(I(PI)^{n-1}x_n-x_n)\right\| \\ &\le (1-\beta_n)\|x_n-x_n\|+\beta_n\left\|I(PI)^{n-1}x_n-x_n\right\| \\ &\le \left\|I(PI)^{n-1}x_n-x_n\right\| \end{aligned}$$

and this implies that

$$\lim_{n\to\infty}\|y_n-x_n\|=0 \tag{3.22}$$

Because T is a uniformly L_1-Lipshitzian mapping, we have

$$\begin{aligned}\left\|x_n-T(PT)^{n-1}x_n\right\| &= \left\|x_n-y_n+y_n-T(PT)^{n-1}x_n\right\| \\ &\le \|x_n-y_n\|+\left\|y_n-T(PT)^{n-1}x_n\right\| \\ &\le \|x_n-y_n\|+\left\|y_n-T(PT)^{n-1}y_n\right\| \\ &+\left\|T(PT)^{n-1}y_n-T(PT)^{n-1}x_n\right\| \\ &\le \|x_n-y_n\|+\left\|y_n-T(PT)^{n-1}y_n\right\|+L\|x_n-y_n\| \\ &\le (1+L)\|x_n-y_n\|+\left\|y_n-T(PT)^{n-1}y_n\right\|. \end{aligned}$$

Therefore, from (3.14) and (3.22) we get

$$\lim_{n\to\infty}\left\|x_n-T(PT)^{n-1}x_n\right\|=0 \tag{3.23}$$

Now, we shall show that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Firstly, we have

$$\begin{aligned}\|x_n - Tx_n\| &\le \left\|x_n - T(PT)^{n-1}x_n\right\| + \left\|T(PT)^{n-1}x_n - T(PT)^{n-1}y_n\right\| \\ &+ \left\|T(PT)^{n-1}y_n - Tx_n\right\| \\ &\le \left\|x_n - T(PT)^{n-1}x_n\right\| + L\|x_n - y_n\| \\ &+ L(\left\|T(PT)^{n-2}y_n - y_n\right\| + \|x_n - y_n\|) \\ &= \left\|x_n - T(PT)^{n-1}x_n\right\| + L\|x_n - y_n\| \\ &+ L\|x_n - y_n\| + L\left\|T(PT)^{n-2}y_n - y_n\right\|.\end{aligned}$$

By using (3.14), (3.22) and (3.23), one can see that

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0.$$

In addition, we have

$$\begin{aligned}\|x_n - Ix_n\| &\le \left\|x_n - I(PI)^{n-1}x_n\right\| + \left\|I(PI)^{n-1}x_n - Ix_n\right\| \\ &\le \left\|x_n - I(PI)^{n-1}x_n\right\| + L\left\|I(PI)^{n-2}x_n - x_n\right\|.\end{aligned}$$

Thus, since I is a uniformly L_2-Lipshitzian mapping and by (3.21) we get

$$\lim_{n\to\infty}\|x_n - Ix_n\| = 0.$$

This completes the proof. □

Now, we give some strong convergence theorems and their proofs.

Theorem 2. *Let E be a uniformly convex real Banach space, K be a nonempty closed convex subset of E which is also a nonexpansive retract with retraction P. Let $T : K \to E$ be a nonself uniformly L_1-Lipshitzian totally asymptotically I-quasi-nonexpansive mappings and $I : K \to K$ be a nonself uniformly L_2-Lipshitzian totally asymptotically quasi-nonexpansive mapping with sequences $\{\mu_n\}, \{\lambda_n\}$ defined by (3.1) such that $\sum_{n=1}^{\infty}\mu_n < \infty, \sum_{n=1}^{\infty}\lambda_n < \infty$. Assume that there exist $M, M^* > 0$ such that $\phi(\lambda) \le M^*\lambda$ for all $\lambda \ge M$. Let $\{x_n\}$ be the sequence as defined (1.9), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[\varepsilon, 1-\varepsilon]$, for some $\varepsilon \in (0,1)$. If T and I satisfy Condition $\left(A'\right)$, then the sequence $\{x_n\}$ converges strongly to a common fixed point of T and I.*

Proof. By Lemma5,

$$\lim_{n\to\infty}\|x_n - Tx_n\| = \lim_{n\to\infty}\|x_n - Ix_n\| = 0.$$

Using Condition $\left(A'\right)$, we get

$$\lim_{n\to\infty} f\left(d\left(x, F\right)\right) = \lim_{n\to\infty}\frac{1}{2}\left(\|x - Tx\| + \|x - Ix\|\right) = 0.$$

Since f is a nondecreasing function and $f(0) = 0$, so it follows that $\lim_{n\to\infty} d\left(x, F\right) = 0$. Now applying the Theorem 1, we obtain the result. □

Theorem 3. *Let E be a uniformly convex real Banach space, K be a nonempty closed convex subset of E which is also a nonexpansive retract with retraction P. Let $T : K \to E$ be a nonself uniformly L_1-Lipshitzian totally asymptotically I-quasi-nonexpansive mappings and $I : K \to K$ be a nonself uniformly L_2-Lipshitzian totally asymptotically quasi-nonexpansive mapping with sequences $\{\mu_n\}, \{\lambda_n\}$ defined by (3.1) such that $\sum_{n=1}^{\infty}\mu_n < \infty, \sum_{n=1}^{\infty}\lambda_n < \infty$. Assume that there exist $M, M^* > 0$ such that $\phi(\lambda) \le M^*\lambda$ for all $\lambda \ge M$. Let $\{x_n\}$ be the sequence as defined (1.9), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[\varepsilon, 1-\varepsilon]$, for some $\varepsilon \in (0,1)$. If one of the mappings T and I is semicompact, then the sequence $\{x_n\}$ converges strongly to a common fixed point of T and I.*

Proof. We may assume that one of the mappings T and I is semicompact. Since $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ and $\lim_{n\to\infty}\|x_n - Ix_n\| = 0$ from Lemma5, then there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \to x^*$ and $x^* \in K$.Thus we get,

$$\lim_{n\to\infty}\|x^* - Tx^*\| = \lim_{n\to\infty}\left\|x_{n_j} - Tx_{n_j}\right\| = 0$$

and

$$\lim_{n\to\infty}\|x^* - Ix^*\| = \lim_{n\to\infty}\left\|x_{n_j} - Ix_{n_j}\right\| = 0.$$

This shows that $x^* \in F$. According to Lemma 4, the limit $\lim_{n\to\infty}\|x_n - x^*\|$ exists. Then

$$\lim_{n\to\infty}\|x_n - x^*\| = \lim_{n\to\infty}\left\|x_{n_j} - x^*\right\| = 0$$

which means that $\{x_n\}$ converges to $x^* \in F$. This completes the proof. □

Finally, we prove our weak convergence theorem.

Theorem 4. *Let E be a uniformly convex real Banach space, K be a nonempty closed convex subset of E which is also a nonexpansive retract with retraction P. Let $T : K \to E$ be a nonself uniformly L_1-Lipshitzian totally asymptotically I-quasi-nonexpansive mappings and $I : K \to K$ be a nonself uniformly L_2-Lipshitzian totally asymptotically quasi-nonexpansive mapping with sequences $\{\mu_n\}, \{\lambda_n\}$ defined by (3.1) such that $\sum_{n=1}^{\infty}\mu_n < \infty, \sum_{n=1}^{\infty}\lambda_n < \infty$. Assume that there exist $M, M^* > 0$ such that $\phi(\lambda) \leq M^*\lambda$ for all $\lambda \geq M$. Let $\{x_n\}$ be the sequence as defined (1.9), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[\varepsilon, 1-\varepsilon]$, for some $\varepsilon \in (0, 1)$. If $E - T$ and $E - I$ are demiclosed at zero then the sequence $\{x_n\}$ converges weakly to a common fixed point of T and I.*

Proof. Let $p \in F$. Then by Lemma 4, $\lim_{n\to\infty}\|x_n - p\|$ exists. We prove that $\{x_n\}$ has a unique weak subsequential limit in F. Since $\{x_n\}$ is a bounded sequence in a uniformly convex Banach space X, there exist two weakly convergent subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$. Let $w_1 \in K$ and $w_2 \in K$ be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$, respectively. Since T is demiclosed with respect to zero (by hypothesis) then we obtain $Tw_1 = w_1$. Similarly, $Iw_1 = w_1$. That is, $w_1 \in F$. In the same way, we can prove that $w_2 \in F$.

Next, we prove the uniqueness. For this, suppose that $w_1 \neq w_2$. Then, by Opial's condition, we have

$$\begin{aligned}
\lim_{n\to\infty}\|x_n - w_1\| &= \lim_{i\to\infty}\|x_{n_i} - w_1\| \\
&< \lim_{i\to\infty}\|x_{n_i} - w_2\| \\
&= \lim_{n\to\infty}\|x_n - w_2\| \\
&= \lim_{j\to\infty}\left\|x_{n_j} - w_2\right\| \\
&< \lim_{j\to\infty}\left\|x_{n_j} - w_1\right\| \\
&= \lim_{n\to\infty}\|x_n - w_1\|,
\end{aligned}$$

which is a contradiction. Hence $\{x_n\}$ converges weakly to a point of F. □

Remark 4. *Since the class of totally asymptotically quasi-I-nonexpansive mappings includes totally asymptotically nonexpansive mappings, our results improve and extend the corresponding ones announced by Ya.I. Alber at. al.* [4], *Chidume and Ofoedu* [5, 6], *Hu and Yang* [7] *to a case of one mapping. Also, our results generalize results given in* [19, 20, 21, 22].

Remark 5. *The iteration process (1.9) can be generalized for a finite families of nonself totally asymptotically I_i-quasi-nonexpansive mappings $\{T_j : j \in J\}$, where $\{I_j : j \in J\}$ is a finite family of nonself totally asymptotically quasi-nonexpansive mappings. (where $J = \{1, 2, ..., N\}$)*

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